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**ON NON-PERTURBATIVE RESULTS
IN SUPERSYMMETRIC GAUGE THEORIES
– A LECTURE¹**

Amit Giveon²

Theory Division, CERN, CH-1211, Geneva 23, Switzerland

ABSTRACT

Some notions in non-perturbative dynamics of supersymmetric gauge theories are being reviewed. This is done by touring through a few examples.

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²Permanent address: Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel; e-mail address: giveon@vms.huji.ac.il

1 Introduction

In this lecture, we present some notions in supersymmetric Yang-Mills (YM) theories. We do it by touring through a few examples where we face a variety of non-perturbative physics effects – infra-red (IR) dynamics of gauge theories.

We shall start with a general review; some of the points we consider follow the beautiful lecture notes in [1].

Phases of Gauge Theories

There are three known phases of gauge theories:

- *Coulomb Phase*: there are massless vector bosons (massless photons γ ; no confinement of both electric and magnetic charges). The behavior of the potential $V(R)$ between electric test charges, separated by a large distance R , is $V(R) \sim 1/R$; the electric charge at large distance behaves like a constant: $e^2(R) \sim \text{constant}$. The potential of magnetic test charges separated by a large distance behaves like $V(R) \sim 1/R$, and the magnetic charge behaves like $m^2(R) \sim \text{constant}$, $e(R)m(R) \sim 1$ (the Dirac condition).
- *Higgs Phase*: there are massive vector bosons (W bosons and Z bosons), electric charges are condensed (screened) and magnetic charges are confined (the Meissner effect). The potential between magnetic test charges separated by a large distance is $V(R) \sim \rho R$ (the magnetic flux is confined into a thin tube, leading to this linear potential with a string tension ρ). The potential between electric test charges is the Yukawa potential; at large distances R it behaves like a constant: $V(R) \sim \text{constant}$.
- *Confining Phase*: magnetic charges are condensed (screened) and electric charges are confined. The potential between electric test charges separated by a large distance is $V(R) \sim \sigma R$ (the electric flux is confined

into a thin tube, leading to the linear potential with a string tension σ). The potential between magnetic test charges behaves like a constant at large distance R .

Remarks

1. In addition to the familiar Abelian Coulomb phase, there are theories which have a *non-Abelian Coulomb phase* [2], namely, a theory with massless interacting quarks and gluons exhibiting the Coulomb potential. This phase occurs when there is a non-trivial IR fixed point of the renormalization group. Such theories are part of other possible cases of non-trivial, *interacting 4d superconformal field theories* (SCFTs) [3, 4].
2. When there are matter fields in the fundamental representation of the gauge group, virtual pairs can be created from the vacuum and screen the sources. In this situation, there is no invariant distinction between the Higgs and the confining phases [5]. In particular, there is no phase with a potential behaving as $V(R) \sim R$ at large distance, because the flux tube can break. For large VEVs of the fields, a Higgs description is most natural, while for small VEVs it is more natural to interpret the theory as “confining.” It is possible to smoothly interpolate from one interpretation to the other.
3. *Electric-Magnetic Duality*: Maxwell theory is invariant under

$$\mathbf{E} \rightarrow \mathbf{B}, \quad \mathbf{B} \rightarrow -\mathbf{E}, \quad (1.1)$$

if we introduce magnetic charge $m = 2\pi/e$ and also interchange

$$e \rightarrow m, \quad m \rightarrow -e. \quad (1.2)$$

Similarly, Mandelstam and ‘t Hooft suggested that under electric-magnetic duality the Higgs phase is interchanged with a confining phase. Confinement can then be understood as the dual Meissner effect associated with a condensate of monopoles.

Dualizing a theory in the Coulomb phase, one remains in the same phase. For an Abelian Coulomb phase with massless photons, this electric-magnetic duality follows from a standard duality transformation, and is extended to $SL(2, \mathbf{Z})$ S-duality, acting on the complex gauge coupling by

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}, \quad a, b, c, d \in \mathbf{Z}, \quad ad - bc = 1. \quad (1.3)$$

In the non-Abelian Coulomb phase this duality is not obvious. The simplest versions of such S-duality in non-trivial interacting theories are in $N = 4$ supersymmetric YM theory [6], and in finite $N = 2$ supersymmetric YM theories [7]. Electric-magnetic duality can be extended, sometimes, to asymptotically free $N = 1$ theories [1].

2 The Low-Energy Effective Theory

We will consider the low-energy effective action for the light fields, \mathcal{L}_{eff} (light fields), integrating out degrees of freedom (such as massive vector bosons, resonances, etc.) above some scale. \mathcal{L}_{eff} has a linearly realized supersymmetry (as long as we are above the scale of possible supersymmetry breaking).

Supersymmetry can be made manifest by working with superfields [8]. All renormalizable Lagrangians can be constructed in terms of chiral (= scalar), anti-chiral and vector supermultiplets:

- *Chiral and Anti-Chiral Superfields*: the light matter fields can be combined into chiral and anti-chiral superfields

$$X_r = \phi_r + \theta_\alpha \psi_r^\alpha + \dots, \quad X_r^\dagger = \phi_r^\dagger + \bar{\theta}_{\dot{\alpha}} \psi_r^{\dagger\dot{\alpha}} + \dots \quad (2.1)$$

Chiral (anti-chiral) superfields obey $\bar{D}X_r = 0$ ($DX_r^\dagger = 0$).

- *Vector Supermultiplet*: the real superfield $V = V^\dagger$ combining the light vector bosons A_μ and the gauginos $\lambda_\alpha, \lambda_\alpha^\dagger$; schematically,

$$V \sim \theta\sigma^\mu\bar{\theta}A_\mu + \theta^2(\bar{\theta}\lambda^\dagger) + \bar{\theta}^2(\theta\lambda) + \dots \quad (2.2)$$

By *effective*, we mean in Wilson sense:

$$\int [\text{modes } p > \mu] e^{-S} = e^{-S_{eff}(\mu, \text{light modes})}, \quad (2.3)$$

so, in principle, \mathcal{L}_{eff} depends on a scale μ . But due to supersymmetry, the dependence on the scale μ disappear (except for the gauge coupling τ which has a $\log \mu$ dependence).

When there are no interacting massless particles, the Wilsonian effective action = the 1PI effective action; this is often the case in the Higgs or confining phases.

2.1 The Effective Superpotential

We will focus on a particular contribution to \mathcal{L}_{eff} – the effective superpotential term:

$$\mathcal{L}_{int} \sim \int d^2\theta W_{eff}(X_r, g_I, \Lambda) + c.c., \quad (2.4)$$

where X_r = light chiral superfields, g_I = various coupling constants, and Λ = dynamically generated scale (associated with the gauge dynamics): $\log(\Lambda/\mu) \sim -8\pi^2/g^2(\mu)$. Integrating over θ , the superpotential gives a scalar potential and Yukawa-type interaction of scalars with fermions.

The quantum, effective superpotential $W_{eff}(X_r, g_I, \Lambda)$ is constrained by holomorphy, global symmetries and various limits [9, 1]:

1. *Holomorphy*: supersymmetry requires that W_{eff} is holomorphic in the chiral superfields X_r (*i.e.*, independent of the X_r^\dagger). Moreover, we will think of all the coupling constants g_I in the tree-level superpotential W_{tree} and the scale Λ as background chiral superfield sources. This implies that W_{eff} is holomorphic in g_I, Λ (*i.e.*, independent of g_I^*, Λ^*).
2. *Symmetries and Selection Rules*: by assigning transformation laws both to the fields and to the coupling constants (which are regarded as background chiral superfields), the theory has a large global symmetry. This implies that W_{eff} should be invariant under such global symmetries.

3. *Various Limits:* W_{eff} can be analyzed approximately at weak coupling, and some other limits (like large masses).

Sometimes, holomorphy, symmetries and various limits are strong enough to determine W_{eff} ! The results can be highly non-trivial, revealing interesting non-perturbative dynamics.

2.2 The Gauge “Kinetic Term” in a Coulomb Phase

When there is a Coulomb phase, there is a term in \mathcal{L}_{eff} of the form

$$\mathcal{L}_{gauge} \sim \int d^2\theta \text{Im}[\tau_{eff}(X_r, g_I, \Lambda) \mathcal{W}_\alpha^2], \quad (2.5)$$

where \mathcal{W}_α = gauge supermultiplet (supersymmetric field strength); schematically, $\mathcal{W}_\alpha \sim \lambda_\alpha + \theta_\beta \sigma_\alpha^{\mu\nu \beta} F_{\mu\nu} + \dots$. Integrating over θ , \mathcal{W}_α^2 gives the term $F^2 + iF\tilde{F}$ and its supersymmetric extension. Therefore,

$$\tau_{eff} = \frac{\theta_{eff}}{2\pi} + i \frac{4\pi}{g_{eff}^2} \quad (2.6)$$

is the effective, complex gauge coupling. $\tau_{eff}(X_r, g_I, \Lambda)$ is also holomorphic in X_r, g_I, Λ and, sometimes, it can be exactly determined by using holomorphy, symmetries and various limits.

2.3 The “Kinetic Term”

The kinetic term is determined by the Kähler potential K :

$$\mathcal{L}_{kin} \sim \int d^2\theta d^2\bar{\theta} K(X_r, X_r^\dagger). \quad (2.7)$$

If there is an $N = 2$ supersymmetry, τ_{eff} and K are related; for an $N = 2$ supersymmetric YM theory with a gauge group G and in a Coulomb phase, \mathcal{L}_{eff} is given in terms of a single holomorphic function $\mathcal{F}(A^i)$:

$$\begin{aligned} \mathcal{L}_{eff} \sim \text{Im} \left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial X^i} X^{\dagger i} + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial X^i \partial X^j} \mathcal{W}^{\alpha i} \mathcal{W}_\alpha^j \right], \\ i, j = 1, \dots, \text{rank} G. \end{aligned} \quad (2.8)$$

A manifestly gauge invariant $N = 2$ supersymmetric action which reduces to the above at low energies is [10]

$$\text{Im} \left[\int d^4\theta \frac{\partial \mathcal{F}}{\partial X^a} (e^V)_{ab} X^{\dagger b} + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}}{\partial X^a \partial X^b} \mathcal{W}^{\alpha a} \mathcal{W}_\alpha^b \right],$$

$$a, b = 1, \dots, \dim G. \quad (2.9)$$

This concludes the general review. Next, we consider some examples of results.

3 Summary of Results in $N = 1$ Supersymmetric $SU(2)$ Gauge Theories

In the next few sections, we shall summarize some results in $4d$, $N = 1$ supersymmetric $SU(2)$ gauge theories: the exact effective superpotentials, the vacuum structure, and the exact effective Abelian couplings for arbitrary bare masses and Yukawa couplings. A few generalizations to other gauge groups will be pointed out.

The Main Result

The results in this summary are based on some of the results in refs. [11, 12].³ We consider $N = 1$ supersymmetric $SU(2)$ gauge theories in four dimensions, with any possible content of matter superfields, such that the theory is either one-loop asymptotic free or conformal. This allows the introduction of $2N_f$ matter supermultiplets in the fundamental representation, Q_i^a , $i = 1, \dots, 2N_f$,

³Some of these results also appear in the proceedings [13] of the *29th International Symposium on the Theory of Elementary Particles* in Buckow, Germany, August 29 - September 2, 1995, and of the workshop on *STU-Dualities and Non-Perturbative Phenomena in Superstrings and Supergravity*, CERN, Geneva, November 27 - December 1, 1995.

N_A supermultiplets in the adjoint representation, Φ_α^{ab} , $\alpha = 1, \dots, N_A$, and $N_{3/2}$ supermultiplets in the spin $3/2$ representation, Ψ . Here a, b are fundamental representation indices, and $\Phi^{ab} = \Phi^{ba}$ (we present Ψ in a schematic form as we shall not use it much). The numbers N_f , N_A and $N_{3/2}$ are limited by the condition:

$$b_1 = 6 - N_f - 2N_A - 5N_{3/2} \geq 0, \quad (3.1)$$

where $-b_1$ is the one-loop coefficient of the gauge coupling beta-function.

The main result of this section is the following: the effective superpotential of an (asymptotically free or conformal) $N = 1$ supersymmetric $SU(2)$ gauge theory, with $2N_f$ doublets and N_A triplets (and $N_{3/2}$ quartets) is

$$\begin{aligned} W_{N_f, N_A}(M, X, Z, N_{3/2}) = & \\ & -\delta_{N_{3/2}, 0}(4 - b_1) \left\{ \Lambda^{-b_1} \text{Pf}_{2N_f} X \left[\det_{N_A}(\Gamma_{\alpha\beta}) \right]^2 \right\}^{1/(4-b_1)} \\ & + \text{Tr}_{N_A} \tilde{m} M + \frac{1}{2} \text{Tr}_{2N_f} m X + \frac{1}{\sqrt{2}} \text{Tr}_{2N_f} \lambda^\alpha Z_\alpha + \delta_{N_{3/2}, 1} g U, \end{aligned} \quad (3.2)$$

where

$$\Gamma_{\alpha\beta}(M, X, Z) = M_{\alpha\beta} + \text{Tr}_{2N_f}(Z_\alpha X^{-1} Z_\beta X^{-1}). \quad (3.3)$$

The first term in (3.2) is the exact (dynamically generated) non-perturbative superpotential⁴, and the other terms form the tree-level superpotential. Λ is the dynamically generated scale, while $\tilde{m}_{\alpha\beta}$, m_{ij} and λ_{ij}^α are the bare masses and Yukawa couplings, respectively ($\tilde{m}_{\alpha\beta} = \tilde{m}_{\beta\alpha}$, $m_{ij} = -m_{ji}$, $\lambda_{ij}^\alpha = \lambda_{ji}^\alpha$). The gauge singlets, X , M , Z , U , are given in terms of the $N = 1$ superfield doublets, Q^a , the triplets, Φ^{ab} , and the quartets, Ψ , as follows:

$$\begin{aligned} X_{ij} &= Q_{ia} Q_j^a, & a = 1, 2, & \quad i, j = 1, \dots, 2N_f, \\ M_{\alpha\beta} &= \Phi_{\alpha b}^a \Phi_{\beta a}^b, & \alpha, \beta = 1, \dots, N_A, & \quad a, b = 1, 2, \\ Z_{ij}^\alpha &= Q_{ia} \Phi_{\alpha b}^a Q_j^b, & & \quad U = \Psi^4. \end{aligned} \quad (3.4)$$

⁴Integrating in the “glueball” field $S = -\mathcal{W}_\alpha^2$, whose source is $\log \Lambda^{b_1}$, gives the non-perturbative superpotential:

$$W(S, M, X, Z) = S \left[\log \left(\frac{\Lambda^{b_1} S^{4-b_1}}{\text{Pf} X (\det \Gamma)^2} \right) - (4 - b_1) \right].$$

Here, the a, b indices are raised and lowered with an ϵ_{ab} tensor. The gauge-invariant superfields X_{ij} may be considered as a mixture of $SU(2)$ “mesons” and “baryons,” while the gauge-invariant superfields Z_{ij}^α may be considered as a mixture of $SU(2)$ “meson-like” and “baryon-like” operators.

Equation (3.2) is a universal representation of the superpotential for all infra-red non-trivial theories; all the physics we shall discuss (and beyond) is in (3.2). In particular, all the symmetries and quantum numbers of the various parameters are already embodied in W_{N_f, N_A} . The non-perturbative superpotential is derived in refs. [11, 12] by an “integrating in” procedure, following refs. [14, 15]. The details can be found in ref. [12] and will not be presented here ⁵. Instead, in the next sections, we list the main results concerning each of the theories, $N_f, N_A, N_{3/2}$, case by case. Moreover, a few generalizations to other gauge groups will be discussed.

4 $b_1 = 6$: $N_f = N_A = N_{3/2} = 0$

This is a pure $N = 1$ supersymmetric $SU(2)$ gauge theory. The non-perturbative effective superpotential is ⁶

$$W_{0,0} = \pm 2\Lambda^3. \quad (4.1)$$

The superpotential in eq. (4.1) is non-zero due to gaugino (gluino) condensation ⁷. Let us consider gaugino condensation for general simple groups [1].

Pure $N = 1$ Supersymmetric Yang-Mills Theories

Pure $N = 1$ supersymmetric gauge theories are theories with pure superglue with no matter. We consider a theory based on a simple group G . The theory

⁵These were discussed in a talk presented by E. Rabinovici in this meeting.

⁶This can be read from eq. (3.2) by setting $\text{Tr}_{2N_f}(\cdot) = 0$, $\det_{2N_f}(\cdot) = 1$ (for example, $\text{Pf}X = 1$, $\Gamma = M$) when $N_f = 0$, and $\text{Tr}_{N_A}(\cdot) = 0$, $\det_{N_A}(\cdot) = 1$ (for example, $\det \Gamma = 1$) when $N_A = 0$; this will also be used later.

⁷Recall that we regard Λ as a background chiral superfield source.

contains vector bosons A_μ and gauginos λ_α in the adjoint representation of G . There is a classical $U(1)_R$ symmetry, gaugino number, which is broken to a discrete \mathbf{Z}_{2C_2} subgroup by instantons,

$$\langle (\lambda\lambda)^{C_2} \rangle = \text{const.} \Lambda^{3C_2}, \quad (4.2)$$

where C_2 = the Casimir in the adjoint representation normalized such that, for example, $C_2 = N_c$ for $G = SU(N_c)$.

This theory confines, gets a mass gap, and there are C_2 vacua associated with the spontaneous breaking of the \mathbf{Z}_{2C_2} symmetry to \mathbf{Z}_2 by gaugino condensation:

$$\langle \lambda\lambda \rangle = \text{const.} e^{2\pi i n / C_2} \Lambda^3, \quad n = 1, \dots, C_2. \quad (4.3)$$

Each of these C_2 vacua contributes $(-)^F = 1$ and thus the Witten index is $\text{Tr}(-)^F = C_2$. This physics is encoded in the generalization of eq. (4.1) to any G , giving

$$W_{eff} = e^{2\pi i n / C_2} C_2 \Lambda^3, \quad n = 1, \dots, C_2. \quad (4.4)$$

For $G = SU(2)$ we have $C_2 = 2$. Indeed, the “ \pm ” in (4.1), which comes from the square-root appearing on the braces in (3.2) when $b_1 = 6$, corresponds, physically, to the two quantum vacua of a pure $N = 1$ supersymmetric $SU(2)$ gauge theory.

The superpotentials (4.1), (4.3) can be derived by first adding fundamental matter to pure $N = 1$ supersymmetric YM theory (as we will do in the next section), and then integrating it out.

5 $b_1 = 5$: $N_f = 1$, $N_A = N_{3/2} = 0$

There is one case with $b_1 = 5$, namely, $SU(2)$ with one flavor. The superpotential is

$$W_{1,0} = \frac{\Lambda^5}{X} + mX, \quad (5.1)$$

where X and m are defined by: $X_{ij} \equiv X\epsilon_{ij}$, $m_{ij} \equiv -m\epsilon_{ij}$. The non-perturbative part of $W_{1,0}$ is proportional to the one instanton action. The

vacuum degeneracy of the classical low-energy effective theory is lifted quantum mechanically; from eq. (5.1) we see that, in the massless case, there is no vacuum at all.

$SU(N_c)$ **with** $N_f < N_c$

Equation (5.1) is a particular case of $SU(N_c)$ with $N_f < N_c$ (N_f quarks Q^i and N_f anti-quarks $\bar{Q}_{\bar{i}}$, $i, \bar{i} = 1, \dots, N_f$) [1]. In these theories, by using holomorphy and global symmetries,

$$\begin{array}{rclcl}
& U(1)_Q & \times & U(1)_{\bar{Q}} & \times & U(1)_R \\
Q : & 1 & & 0 & & 0 \\
\bar{Q} : & 0 & & 1 & & 0 \\
\Lambda^{3N_c-N_f} : & N_f & & N_f & & 2N_c - 2N_f \\
W : & 0 & & 0 & & 2
\end{array} \tag{5.2}$$

one finds that

$$W_{eff} = (N_c - N_f) \left(\frac{\Lambda^{3N_c-N_f}}{\det X} \right)^{\frac{1}{N_c-N_f}}, \tag{5.3}$$

where

$$X_{\bar{i}}^i \equiv Q^i \bar{Q}_{\bar{i}}, \quad i, \bar{i} = 1, \dots, N_f. \tag{5.4}$$

Classically, $SU(N_c)$ with $N_f < N_c$ is broken down to $SU(N_c - N_f)$. The effective superpotential in (5.3) is dynamically generated by gaugino condensation in $SU(N_c - N_f)$ (for $N_f \leq N_c - 2$)⁸, and by instantons (for $N_f = N_c - 1$).

The $SU(2)$ with $N_f = 1$ Example

For example, let us elaborate on the derivation and physics of eq. (5.1). An $SU(2)$ effective theory with two doublets Q_i^a has one light degree of freedom: four Q_i^a ($i = 1, 2$ is a flavor index, $a = 1, 2$ is a color index; $2 \times 2 = 4$) three

⁸This is the reason to the fractional $1/(N_c - N_f)$ power in (5.3) leading to $C_2(SU(N_c - N_f)) = N_c - N_f$ different phases in W_{eff} .

out of which are eaten by $SU(2)$, leaving $4 - 3 = 1$. This single light degree of freedom can be described by the gauge singlet

$$X = Q_1 Q_2. \quad (5.5)$$

When $\langle X \rangle \neq 0$, $SU(2)$ is completely broken and, classically, $W_{eff,class} = 0$ (when $\langle X \rangle = 0$ there are extra massless fields due to an unbroken $SU(2)$). Therefore, the classical scalar potential is identically zero. However, the one-instanton action is expected to generate a non-perturbative superpotential.

The symmetries of the theory (at the classical level and with their corresponding charges) are: $U(1)_{Q_1}$ = number of Q_1 fields (quarks or squarks), $U(1)_{Q_2}$ = number of Q_2 fields (quarks or squarks), $U(1)_R = \{\text{number of gluinos}\} - \{\text{number of squarks}\}$. At the quantum level these symmetries are anomalous – $\partial_\mu j^\mu \sim F\tilde{F}$ – and by integrating both sides of this equation one gets a charge violation when there is an instanton background I . The instanton background behaves like

$$I \sim e^{-8\pi^2/g^2(\mu)} = \left(\frac{\Lambda}{\mu}\right)^{b_1}, \quad b_1 = 5, \quad (5.6)$$

and it has four gluino zero-modes λ and two squark zero-modes q . Therefore, Λ^5 has an R -charge = $4(\lambda) - 2(q) = 2$,⁹ and we can summarize the quantum numbers of X and Λ :

$$\begin{array}{ccc} & U(1)_{Q_1} & \times & U(1)_{Q_2} & \times & U(1)_R \\ X : & 1 & & 1 & & 0 \\ \Lambda^5 : & 1 & & 1 & & 2 \end{array} \quad (5.7)$$

Moreover, $\mathcal{L}_{eff} = \int d^2\theta W_{eff}(X, \Lambda)$ has charges = 0, while $d^2\theta$ has charges:

$$d^2\theta : \quad \begin{array}{ccc} U(1)_{Q_1} & \times & U(1)_{Q_2} & \times & U(1)_R \\ 0 & & 0 & & -2 \end{array} \quad (5.8)$$

⁹For $SU(N_c)$ with N_f flavors, the instanton background I has $2C_2 = 2N_c$ gluino zero-modes λ and $2N_f$ squark zero-modes q and, therefore, its R -charge is $R(I) = \text{number}(\lambda) - \text{number}(q) = 2N_c - 2N_f$. Since $I \sim \Lambda^{b_1}$ and $b_1 = 3N_c - N_f$, we learn that $\Lambda^{3N_c - N_f}$ has an R -charge = $2N_c - 2N_f$, as it appears in eq. (5.2).

and, therefore, W_{eff} has charges:

$$W_{eff} : \begin{array}{ccc} U(1)_{Q_1} & \times & U(1)_{Q_2} & \times & U(1)_R \\ 0 & & 0 & & 2 \end{array} \quad (5.9)$$

Finally, because W_{eff} is holomorphic in X, Λ , and is invariant under symmetries, we must have

$$W_{eff}(X, \Lambda) = c \frac{\Lambda^5}{X}. \quad (5.10)$$

This is an exact result. To determine the constant c , one needs to do the one-instanton calculation. The calculation is well defined, and done by Affleck, Dine and Seiberg [16] with the result: $c \neq 0$. We can choose units for Λ (or a renormalization scheme) to scale c to 1. Therefore, as was advocated, the vacuum degeneracy of the classical low-energy effective theory is lifted quantum mechanically: the scalar potential is large at strong coupling (small $|X|$) and goes asymptotically to 0 at large $|X|$ (as it should at weak coupling).

6 $b_1 = 4$

There are two cases with $b_1 = 4$: either $N_f = 2$, or $N_A = 1$. In both cases, the non-perturbative superpotential vanishes and, in addition, there is a constraint¹⁰.

6.1 $N_f = 2, N_A = N_{3/2} = 0$

The non-perturbative superpotential vanishes

$$W_{2,0}^{non-per.} = 0, \quad (6.1)$$

and by the integrating in procedure we also get the quantum constraint:

$$\text{Pf} X = \Lambda^4. \quad (6.2)$$

¹⁰This is reflected in eq. (3.2) by the vanishing of the coefficient $(4 - b_1)$ in front of the braces, leading to $W = 0$, and the singular power $1/(4 - b_1)$ on the braces, when $b_1 = 4$, which signals the existence of a constraint.

At the classical limit, $\Lambda \rightarrow 0$, the quantum constraint collapses into the classical constraint, $\text{Pf}X = 0$.

$$SU(N_c) \text{ with } N_f = N_c$$

Equations (6.1), (6.2) are a particular case of $SU(N_c)$ with $N_f = N_c$. [1] In these theories one obtains $W_{eff} = 0$, and the classical constraint $\det X - B\bar{B} = 0$ is modified quantum mechanically to

$$\det X - B\bar{B} = \Lambda^{2N_c}, \quad (6.3)$$

where

$$\begin{aligned} X_{\bar{i}}^i &= Q^i \bar{Q}_{\bar{i}} \quad (\text{mesons}), \\ B &= \epsilon_{i_1 \dots i_{N_c}} Q^{i_1} \dots Q^{i_{N_c}} \quad (\text{baryon}), \\ \bar{B} &= \epsilon^{\bar{i}_1 \dots \bar{i}_{N_c}} \bar{Q}_{\bar{i}_1} \dots \bar{Q}_{\bar{i}_{N_c}} \quad (\text{anti-baryon}). \end{aligned} \quad (6.4)$$

6.2 $N_f = 0, N_A = 1, N_{3/2} = 0$

The massless $N_A = 1$ case is a pure $SU(2)$, $N = 2$ supersymmetric Yang-Mills theory. This model was considered in detail in ref. [17]. The non-perturbative superpotential vanishes

$$W_{0,1}^{non-per.} = 0, \quad (6.5)$$

and by the integrating in procedure we also get the quantum constraint:

$$M = \pm \Lambda^2. \quad (6.6)$$

This result can be understood because the starting point of the integrating in procedure is a pure $N = 1$ supersymmetric Yang-Mills theory. Therefore, it leads us to the points at the verge of confinement in the moduli space. These are the two singular points in the M moduli space of the theory; they are due to massless monopoles or dyons. Such excitations are not constructed out of the elementary degrees of freedom and, therefore, there is no trace for them in W . (This situation is different if $N_f \neq 0$, $N_A = 1$; in this case, monopoles are different manifestations of the elementary degrees of freedom.)

7 $b_1 = 3$

There are two cases with $b_1 = 3$: either $N_f = 3$, or $N_A = N_f = 1$. In both cases, for vanishing bare parameters in (3.2), the semi-classical limit, $\Lambda \rightarrow 0$, imposes the classical constraints, given by the equations of motion: $\partial W = 0$; however, quantum corrections remove the constraints.

7.1 $N_f = 3, N_A = N_{3/2} = 0$

The superpotential is

$$W_{3,0} = -\frac{\text{Pf}X}{\Lambda^3} + \frac{1}{2}\text{Tr}mX. \quad (7.1)$$

In the massless case, the equations $\partial_X W = 0$ give the classical constraints; in particular, the superpotential is proportional to a classical constraint: $\text{Pf}X = 0$. The negative power of Λ , in eq. (7.1) with $m = 0$, indicates that small values of Λ imply a semi-classical limit for which the classical constraints are imposed.

$$SU(N_c) \text{ with } N_f = N_c + 1$$

Equation (7.1) is a particular case of $SU(N_c)$ with $N_f = N_c + 1$ [1]. In these theories one obtains

$$W_{eff} = -\frac{\det X - X_i^i B_i \bar{B}^{\bar{i}}}{\Lambda^{2N_c-1}}, \quad (7.2)$$

where

$$\begin{aligned} X_i^i &= Q^i \bar{Q}_{\bar{i}} \quad (\text{mesons}), \\ B_i &= \epsilon_{ij_1 \dots j_{N_c}} Q^{j_1} \dots Q^{j_{N_c}} \quad (\text{baryons}), \\ \bar{B}^{\bar{i}} &= \epsilon^{\bar{i}\bar{j}_1 \dots \bar{j}_{N_c}} \bar{Q}_{\bar{j}_1} \dots \bar{Q}_{\bar{j}_{N_c}} \quad (\text{anti-baryons}). \end{aligned} \quad (7.3)$$

The equations $\partial_X W_{eff} = \partial_B W_{eff} = \partial_{\bar{B}} W_{eff} = 0$ give the classical constraints:

$$\det X (X^{-1})_i^{\bar{i}} - B_i \bar{B}^{\bar{i}} = X_i^i B_i = X_i^i \bar{B}^{\bar{i}} = 0. \quad (7.4)$$

This is consistent with the negative power of Λ in W_{eff} which implies that in the semi-classical limit, $\Lambda \rightarrow 0$, the classical constraints are imposed.

7.2 $N_f = 1, N_A = 1, N_{3/2} = 0$

In this case, the superpotential in (3.2) reads

$$W_{1,1} = -\frac{\text{Pf}X}{\Lambda^3}\Gamma^2 + \tilde{m}M + \frac{1}{2}\text{Tr}mX + \frac{1}{\sqrt{2}}\text{Tr}\lambda Z. \quad (7.5)$$

Here m, X are antisymmetric 2×2 matrices, λ, Z are symmetric 2×2 matrices and

$$\Gamma = M + \text{Tr}(ZX^{-1})^2. \quad (7.6)$$

This superpotential was found first in ref. [18]. To find the quantum vacua, we solve the equations: $\partial_M W = \partial_X W = \partial_Z W = 0$. Let us discuss some properties of this theory:

- The equations $\partial W = 0$ can be re-organized into the singularity conditions of an elliptic curve:

$$y^2 = x^3 + ax^2 + bx + c \quad (7.7)$$

(and some other equations), where the coefficients a, b, c are functions of only the field M , the scale Λ , the bare quark masses, m , and Yukawa couplings, λ . Explicitly,

$$a = -M, \quad b = \frac{\Lambda^3}{4}\text{Pf}m, \quad c = -\frac{\alpha}{16}, \quad (7.8)$$

where

$$\alpha = \frac{\Lambda^6}{4}\det\lambda. \quad (7.9)$$

- The parameter x , in the elliptic curve (7.7), is given in terms of the composite field:

$$x \equiv \frac{1}{2}\Gamma. \quad (7.10)$$

- $W_{1,1}$ has $2 + N_f = 3$ vacua, namely, the three singularities of the elliptic curve in (7.7), (7.8). These are the three solutions, $M(x)$, of the equations: $y^2 = \partial y^2 / \partial x = 0$; the solutions for X, Z are given by the other equations of motion.
- The 3 *quantum vacua* are the vacua of the theory in the *Higgs-confinement phase*.
- Phase transition points to the *Coulomb branch* are at $X = 0 \Leftrightarrow \tilde{m} = 0$. Two of these singularities correspond to a massless monopole or dyon, and are the quantum splitting of the classically enhanced $SU(2)$ point. A third singularity is due to a massless quark; it is a classical singularity: $M \sim m^2 / \lambda^2$ for large m , and thus $M \rightarrow \infty$ when $m \rightarrow \infty$, leaving the two quantum singularities of the $N_A = 1, N_f = 0$ theory.
- The elliptic curve defines the *effective Abelian coupling*, $\tau(M, \Lambda, m, \lambda)$, in the Coulomb branch:

Elliptic Curves and Effective Abelian Couplings

A torus can be described by the one complex dimensional curve in \mathbf{C}^2 $y^2 = x^3 + ax^2 + bx + c$, where $(x, y) \in \mathbf{C}^2$ and a, b, c are complex parameters. The modular parameter of the torus is

$$\tau(a, b, c) = \frac{\int_{\beta} \frac{dx}{y}}{\int_{\alpha} \frac{dx}{y}}, \quad (7.11)$$

where α and β refer to a basis of cycles around the branch cuts of the curve in the x plan.

Alternatively, by redefining $x \rightarrow x - a/3$, we can take the curve into the normal Weierstrass form

$$y^2 = x^3 + fx + g. \quad (7.12)$$

In this form, the modular parameter τ is determined (modulo $SL(2, \mathbf{Z})$) by the ratio f^3/g^2 through the relation

$$j(\tau) = \frac{4(24f)^3}{4f^3 + 27g^2}, \quad (7.13)$$

where j is the modular-invariant holomorphic function [19]. The singularities of the curve are located at the zeros of the discriminant

$$\Delta = 4f^3 + 27g^2. \quad (7.14)$$

Therefore, eq. (7.7) defines an $SL(2, \mathbf{Z})$ section τ in terms of M , various coupling constants and a scale, which is singular at the zeroes of Δ .

- On the subspace of bare parameters, where the theory has an enhanced $N = 2$ supersymmetry, the result in eq. (7.8) coincides with the result in [7] for $N_f = 1$.
- In the massless case, there is a $Z_{4-N_f} = Z_3$ global symmetry acting on the moduli space.
- When the masses and Yukawa couplings approach zero, all the 3 singularities collapse to the origin. Such a point might be interpreted as a new scale-invariant theory [1]. As before, the negative power of Λ , in eq. (7.5) with $\tilde{m} = m = \lambda = 0$, indicates that small values of Λ imply a semi-classical limit for which the classical constraint, $\Gamma = 0$, is imposed. Indeed, for vanishing bare parameters, the equations of motion are solved by any M, X, Z obeying $\Gamma = 0$.
- By tuning the quark mass m , one can find a situation where two out of the three singularities degenerate [4]. At the phase transition to the Coulomb branch, this theory has massless, mutually non-local charged degrees of freedom; it is a new non-trivial, interacting $N = 1$ SCFT.

8 $b_1 = 2$

There are three cases with $b_1 = 2$: $N_f = 4$, or $N_A = 1$, $N_f = 2$, or $N_A = 2$. In all three cases, for vanishing bare parameters in (3.2), there are extra massless degrees of freedom not included in the procedure; those are expected due to a non-Abelian conformal theory.

8.1 $N_f = 4$, $N_A = N_{3/2} = 0$

The superpotential is

$$W_{4,0} = -2 \frac{(\text{Pf} X)^{\frac{1}{2}}}{\Lambda} + \frac{1}{2} \text{Tr} m X. \quad (8.1)$$

This theory is a particular case of $SU(N_c)$ with $N_f > N_c + 1$, considered in ref. [1]. In the massless case, the superpotential is proportional to the square-root of a classical constraint: $\text{Pf} X = 0$. The branch cut at $\text{Pf} X = 0$ signals the appearance of extra massless degrees of freedom at these points. Therefore, we make use of the superpotential only in the presence of masses, m , which fix the vacua away from such points.

Non-Abelian Superconformal Field Theories and Duality

To describe this phenomenon of appearance of extra interacting, massless degrees of freedom at $\langle X \rangle = 0$ in eq. (8.1), we shall discuss briefly $SU(N_c)$ with N_f flavors IR theories, which are non-trivial $4d$ superconformal field theories (SCFTs), and the Seiberg duality [2] which “predicts,” in particular, the appearance of interacting, massless degrees of freedoms at the points where the superpotential develops a branch cut.

- *Non-Abelian Coulomb Phase*: there is a strong evidence that for $SU(N_c)$ with $\frac{3}{2}N_c < N_f < 3N_c$ the theory is in an interacting, non-Abelian Coulomb phase (in the IR and for $m = 0$). In this range of N_f the theory is asymptotically free. Namely, at short distance the coupling

constant g is small, and it becomes larger at larger distance. However, it is argued that for $\frac{3}{2}N_c < N_f < 3N_c$, g does not grow to infinity, but it reaches a finite value – a fixed point of the renormalization group. Therefore, for $\frac{3}{2}N_c < N_f < 3N_c$, the IR theory is a non-trivial $4d$ SCFT. The elementary quarks and gluons are not confined but appear as interacting massless particles. The potential between external massless electric sources behaves as $V \sim 1/R$, and thus one refers to this phase of the theory as the non-Abelian Coulomb phase.

- *The Seiberg Duality*: it is claimed [2] that in the IR an $SU(N_c)$ theory with N_f flavors is dual to $SU(N_f - N_c)$ with N_f flavors but, in addition to dual quarks, one should also include interacting, massless scalars. This is the origin to the branch cut in W_{eff} at $\langle X \rangle = 0$, because W_{eff} does not include these light modes which must appear at $\langle X \rangle = 0$.

The quantum numbers of the quarks and anti-quarks of the $SU(N_c)$ theory with N_f flavors (= theory A) are ¹¹

A. $SU(N_c)$, N_f : The Electric Theory

$$\begin{array}{cccccc}
& SU(N_f)_L & \times & SU(N_f)_R & \times & U(1)_B & \times & U(1)_R \\
Q : & N_f & & 1 & & 1 & & 1 - \frac{N_c}{N_f} \\
\bar{Q} : & 1 & & \bar{N}_f & & -1 & & 1 - \frac{N_c}{N_f}
\end{array} \quad (8.2)$$

The quantum numbers of the dual quarks q_i and anti-quarks \bar{q}^i of the $SU(N_f - N_c)$ theory with N_f flavors theory (= theory B) and its massless scalars $X_{\bar{i}}^i$ are

B. $SU(N_f - N_c)$, N_f : The Magnetic Theory

¹¹Here we use the convention where the axial $U(1)_A$ symmetry is anomalous while the R -charge is conserved; previously, we used the opposite convention.

$$\begin{array}{ccccccc}
& SU(N_f)_L & \times & SU(N_f)_R & \times & U(1)_B & \times & U(1)_R \\
q : & \bar{N}_f & & 1 & & \frac{N_c}{N_f - N_c} & & \frac{N_c}{N_f} \\
\bar{q} : & 1 & & N_f & & -\frac{N_c}{N_f - N_c} & & \frac{N_c}{N_f} \\
X : & N_f & & \bar{N}_f & & 0 & & 2 \left(1 - \frac{N_c}{N_f}\right)
\end{array} \quad (8.3)$$

The $U(1)_R$ numbers in both theory A and theory B are the non-anomalous R -charges. The baryon numbers (under $U(1)_B$) of the dual quarks and anti-quarks are designed to construct dual baryons and anti-baryons with B -charge equal to the B -charge of the baryons and anti-baryons in theory A. We notice that these baryon numbers are fractional and, therefore, the dual quarks cannot be represented by a polynomial in the original quarks Q and antiquarks \bar{Q} ; in terms of theory A, they are some (non-local, composite) magnetic degrees of freedom. Thus, while we refer to theory A as the “electric theory,” we call its dual the “magnetic theory,” and the duality relating theory A to theory B is an electric-magnetic duality.

The massless scalars in theory B are independent degrees of freedom ($X \neq q\bar{q}$ except for the self-dual case: $N_f = 2N_c$). Moreover, one should add the tree-level superpotential

$$W_{tree} = X_i^i q_i \bar{q}^{\bar{i}}, \quad (8.4)$$

and, therefore, these massless scalars are interacting, as advocated.

- *Non-Trivial Checks of Duality:*
 1. *‘t Hooft Anomaly Matching Condition:* it says that if two theories are equivalent they must have the same *global* symmetries and, therefore, if we make this global symmetry local in both theories we must obtain the same anomaly numbers with respect to such a gauge symmetry. It was checked [2] that, miraculously, both

theory A and theory B have the same anomalies:

$$\begin{aligned}
U(1)_B^3 &: & 0 \\
U(1)_B U(1)_R^2 &: & 0 \\
U(1)_B^2 U(1)_R &: & -2N_c^2 \\
SU(N_f)^3 &: & N_c d^{(3)}(N_f) \\
SU(N_f)^2 U(1)_R &: & -\frac{N_c^2}{N_f} d^{(2)}(N_f) \\
SU(N_f)^2 U(1)_B &: & N_c d^{(2)}(N_f) \\
U(1)_R &: & -N_c^2 - 1 \\
U(1)_R^3 &: & N_c^2 - 1 - 2\frac{N_c^4}{N_f^2}
\end{aligned} \tag{8.5}$$

Here $d^{(3)}(N_f) = \text{Tr} T_f^3$ of the global $SU(N_f)$ symmetries, where T_f are generators in the fundamental representation, and $d^{(2)}(N_f) = \text{Tr} T_f^2$

2. *Deformations:* theory A and theory B have the same quantum moduli space of deformations.

Remarks

- Electric-magnetic duality exchanges strong coupling with weak coupling (this can be read off from the beta-functions), and it interchanges a theory in the Higgs phase with a theory in the confining phase.
- Strong-weak coupling duality also relates an $SU(N_c)$ theory with $N_f \geq 3N_c$ to an $SU(N_f - N_c)$ theory. $SU(N_c)$ with $N_f \geq 3N_c$ is in a non-Abelian *free electric phase*: in this range the theory is not asymptotically free. Namely, because of screening, the coupling constant becomes smaller at large distance. Therefore, the spectrum of the theory at large distance can be read off from the Lagrangian – it consists of the elementary quarks and gluons. The long distance behavior of the potential between external electric test charges is

$$V(R) \sim \frac{1}{R \log(R\Lambda)} \sim \frac{e^2(R)}{R}, \quad e(R \rightarrow \infty) \rightarrow 0. \tag{8.6}$$

For $N_f \geq 3N_c$, the theory is thus in a non-Abelian free electric phase; the massless electrically charged fields renormalize the charge to zero

at long distance as $e^{-2}(R) \sim \log(R\Lambda)$. The potential of magnetic test charges behave at large distance R as

$$V(R) \sim \frac{\log(R\Lambda)}{R} \sim \frac{m^2(R)}{R}, \quad \Rightarrow \quad e(R)m(R) \sim 1. \quad (8.7)$$

$SU(N_c)$ with $N_f \geq 3N_c$ is dual to $SU(\tilde{N}_c)$ with $\tilde{N}_c + 2 \leq N_f \leq \frac{3}{2}\tilde{N}_c$, where $\tilde{N}_c = N_f - N_c$. This dual theory is in a non-Abelian *free magnetic phase*: there are massless magnetically charged fields X, q, \bar{q} and a potential between external electric test charges with a conjectured behavior at large distance as

$$V(R) \sim \frac{\log(R\Lambda)}{R} \sim \frac{e^2(R)}{R}, \quad e(R \rightarrow \infty) \rightarrow \infty. \quad (8.8)$$

For $\tilde{N}_c + 2 \leq N_f \leq \frac{3}{2}\tilde{N}_c$, the massless magnetic monopoles renormalize the electric coupling constant to infinity at large distance, with a conjectured behavior $e^2(R) \sim \log(R\Lambda)$. The potential of magnetic test charges behaves at large distance R as

$$V(R) \sim \frac{1}{R \log(R\Lambda)} \sim \frac{m^2(R)}{R} \quad \Rightarrow \quad e(R)m(R) \sim 1. \quad (8.9)$$

- The Seiberg duality can be generalized in many other cases, including a variety of matter supermultiplets (like superfields in the adjoint representation [20]) and other gauge groups [21].

8.2 $N_f = 2, N_A = 1, N_{3/2} = 0$

In this case, the superpotential in (3.2) reads

$$W_{2,1} = -2 \frac{(\text{Pf} X)^{\frac{1}{2}}}{\Lambda} \Gamma + \tilde{m} M + \frac{1}{2} \text{Tr} m X + \frac{1}{\sqrt{2}} \text{Tr} \lambda Z. \quad (8.10)$$

Here m, X are antisymmetric 4×4 matrices, λ, Z are symmetric 4×4 matrices and Γ is given in eq. (7.6). As in section 7.2, to find the quantum vacua, we solve the equations: $\partial W = 0$. Let us discuss some properties of this theory:

- The equations $\partial W = 0$ can be re-organized into the singularity conditions of an elliptic curve (7.7) (and some other equations), where the coefficients a, b, c are functions of only the field M , the scale Λ , the bare quark masses, m , and Yukawa couplings, λ . Explicitly [11, 12],

$$a = -M, \quad b = -\frac{\alpha}{4} + \frac{\Lambda^2}{4} \text{Pf} m, \quad c = \frac{\alpha}{8} (2M + \text{Tr}(\mu^2)), \quad (8.11)$$

where

$$\alpha = \frac{\Lambda^4}{16} \det \lambda, \quad \mu = \lambda^{-1} m. \quad (8.12)$$

- As in section 7.2, the parameter x , in the elliptic curve (7.7), is given in terms of the composite field:

$$x \equiv \frac{1}{2} \Gamma. \quad (8.13)$$

Therefore, we have identified a physical meaning of the parameter x .

- $W_{2,1}$ has $2 + N_f = 4$ vacua, namely, the four singularities of the elliptic curve in (7.7), (8.11). These are the four solutions, $M(x)$, of the equations: $y^2 = \partial y^2 / \partial x = 0$; the solutions for X, Z are given by the other equations of motion.
- The *4 quantum vacua* are the vacua of the theory in the *Higgs-confinement phase*.
- Phase transition points to the *Coulomb branch* are at $X = 0 \Rightarrow \tilde{m} = 0$. Therefore, we conclude that the elliptic curve defines the *effective Abelian coupling*, $\tau(M, \Lambda, m, \lambda)$, in the Coulomb branch.
- On the subspace of bare parameters, where the theory has an enhanced $N = 2$ supersymmetry, the result in eq. (8.11) coincides with the result in [7] for $N_f = 2$.
- In the massless case, there is a $Z_{4-N_f} = Z_2$ global symmetry acting on the moduli space.

- As in section 7.2, the negative power of Λ , in eq. (8.10) with $\tilde{m} = m = \lambda = 0$, indicates that small values of Λ imply a semi-classical limit for which the classical constraints are imposed. Indeed, for vanishing bare parameters, the equations $\partial W = 0$ are equivalent to the classical constraints, and their solutions span the Higgs moduli space [22].
- For special values of the bare masses and Yukawa couplings, some of the 4 vacua degenerate. In some cases, it may lead to points where mutually non-local degrees of freedom are massless, similar to the situation in pure $N = 2$ supersymmetric gauge theories, considered in [3]. For example, when the masses and Yukawa couplings approach zero, all the 4 singularities collapse to the origin. Such points might be interpreted as in a *non-Abelian Coulomb phase* [1] or new non-trivial, interacting, $N = 1$ SCFTs.
- The singularity at $X = 0$ (in Γ) and the branch cut at $\text{Pf} X = 0$ (due to the $1/2$ power in eq. (8.10)) signal the appearance of extra massless degrees of freedom at these points; those are expected similar to references [2, 20]. Therefore, we make use of the superpotential only in the presence of bare parameters, which fix the vacua away from such points.

8.3 $N_f = 0$, $N_A = 2$, $N_{3/2} = 0$

In this case, the superpotential in eq. (3.2) reads

$$W_{0,2} = \pm 2 \frac{\det M}{\Lambda} + \text{Tr} \tilde{m} M. \quad (8.14)$$

Here \tilde{m} , M are 2×2 symmetric matrices, and the “ \pm ” comes from the square-root, appearing on the braces in (3.2), when $b_1 = 2$. The superpotential in eq. (8.14) is the one presented in [18, 23] on the confinement and the oblique confinement branches ¹² (they are related by a discrete symmetry [1]). This

¹² The fractional power $1/(4 - b_1)$ on the braces in (3.2), for any theory with $b_1 \leq 2$, may indicate a similar phenomenon, namely, the existence of confinement and oblique

theory has *two quantum vacua*; these become the phase transition points to the Coulomb branch when $\det \tilde{m} = 0$. The moduli space may also contain a non-Abelian Coulomb phase when the two singularities degenerate at the point $M = 0$ [18]; this happens when $\tilde{m} = 0$. At this point, the theory has extra massless degrees of freedom and, therefore, $W_{0,2}$ fails to describe the physics at $\tilde{m} = 0$. Moreover, at $\tilde{m} = 0$, the theory has other descriptions via an electric-magnetic triality [1].

9 $b_1 = 1$

There are four cases with $b_1 = 1$: $N_f = 5$, or $N_A = 1$, $N_f = 3$, or $N_A = 2$, $N_f = 1$, or $N_{3/2} = 1$.

9.1 $N_f = 5, N_A = N_{3/2} = 0$

The superpotential is

$$W_{5,0} = -3 \frac{(\text{Pf} X)^{\frac{1}{3}}}{\Lambda^{\frac{1}{3}}} + \frac{1}{2} \text{Tr} m X. \quad (9.1)$$

This theory is a particular case of $SU(N_c)$ with $N_f > N_c + 1$. The discussion in section 8.1 is relevant in this case too.

9.2 $N_f = 3, N_A = 1, N_{3/2} = 0$

In this case, the superpotential in (3.2) reads

$$W_{3,1} = -3 \frac{(\text{Pf} X)^{\frac{1}{3}}}{\Lambda^{\frac{1}{3}}} \Gamma^{\frac{2}{3}} + \tilde{m} M + \frac{1}{2} \text{Tr} m X + \frac{1}{\sqrt{2}} \text{Tr} \lambda Z. \quad (9.2)$$

Here m, X are antisymmetric 6×6 matrices, λ, Z are symmetric 6×6 matrices and Γ is given in eq. (7.6). Let us discuss some properties of this theory:

confinement branches of the theory, corresponding to the $4 - b_1$ phases due to the fractional power. It is plausible that, for $SU(2)$, such branches are related by a discrete symmetry.

- The equations $\partial W = 0$ can be re-organized into the singularity conditions of an elliptic curve (7.7) (and some other equations), where the coefficients a, b, c are [11, 12]

$$\begin{aligned} a &= -M - \alpha, & b &= 2\alpha M + \frac{\alpha}{2}\text{Tr}(\mu^2) + \frac{\Lambda}{4}\text{Pf}m, \\ c &= \frac{\alpha}{8}\left(-8M^2 - 4M\text{Tr}(\mu^2) - [\text{Tr}(\mu^2)]^2 + 2\text{Tr}(\mu^4)\right), \end{aligned} \quad (9.3)$$

where

$$\alpha = \frac{\Lambda^2}{64} \det \lambda, \quad \mu = \lambda^{-1}m. \quad (9.4)$$

In eq. (9.3) we have shifted the quantum field M to

$$M \rightarrow M - \alpha/4. \quad (9.5)$$

- The parameter x , in the elliptic curve (7.7), is given in terms of the composite field:

$$x \equiv \frac{1}{2}\Gamma + \frac{\alpha}{2}. \quad (9.6)$$

Therefore, as before, we have identified a physical meaning of the parameter x .

- $W_{3,1}$ has $2+N_f = 5$ *quantum vacua*, corresponding to the 5 singularities of the elliptic curve (7.7), (9.3); these are the vacua of the theory in the *Higgs-confinement phase*.
- From the phase transition points to the *Coulomb branch*, we conclude that the elliptic curve defines the *effective Abelian coupling*, $\tau(M, \Lambda, m, \lambda)$, for arbitrary bare masses and Yukawa couplings. As before, on the subspace of bare parameters, where the theory has $N = 2$ supersymmetry, the result in eq. (9.3) coincides with the result in [7] for $N_f = 3$.
- For special values of the bare masses and Yukawa couplings, some of the 5 vacua degenerate. In some cases, it may lead to points where mutually non-local degrees of freedom are massless, and might be interpreted as in a *non-Abelian Coulomb phase* or another new *superconformal theory* in four dimensions (see the discussion in sections 7.2 and 8.2).

- The singularity and branch cuts in $W_{3,1}$ signal the appearance of extra massless degrees of freedom at these points.
- The discussion in the end of sections 7.2 and 8.2 is relevant here too.

9.3 $N_f = 1, N_A = 2, N_{3/2} = 0$

In this case, the superpotential in (3.2) reads [12]

$$W_{1,2} = -3 \frac{(\text{Pf} X)^{1/3}}{\Lambda^{1/3}} (\det \Gamma)^{2/3} + \text{Tr} \tilde{m} M + \frac{1}{2} \text{Tr} m X + \frac{1}{\sqrt{2}} \text{Tr} \lambda^\alpha Z_\alpha. \quad (9.7)$$

Here m and X are antisymmetric 2×2 matrices, λ^α and Z_α are symmetric 2×2 matrices, $\alpha = 1, 2$, \tilde{m} , M are 2×2 symmetric matrices and $\Gamma_{\alpha\beta}$ is given in eq. (3.3). This theory has $\mathcal{3}$ *quantum vacua* in the Higgs-confinement branch. At the phase transition points to the Coulomb branch, namely, when $\det \tilde{m} = 0 \Leftrightarrow \det M = 0$, the equations of motion can be re-organized into the singularity conditions of an elliptic curve (7.7). Explicitly, when $\tilde{m}_{22} = \tilde{m}_{12} = 0$, the coefficients a, b, c in (7.7) are [12]

$$a = -M_{22}, \quad b = \frac{\Lambda \tilde{m}_{11}^2}{16} \text{Pf} m, \quad c = -\left(\frac{\Lambda \tilde{m}_{11}^2}{32}\right)^2 \det \lambda_2. \quad (9.8)$$

However, unlike the $N_A = 1$ cases, the equations $\partial W = 0$ *cannot* be re-organized into the singularity condition of an elliptic curve, in general. This result makes sense, physically, since an elliptic curve is expected to “show up” only at the phase transition points to the Coulomb branch. For special values of the bare parameters, there are points in the moduli space where (some of) the singularities degenerate; such points might be interpreted as in a non-Abelian Coulomb phase, or new superconformal theories. For more details, see ref. [12].

9.4 $N_f = N_A = 0, N_{3/2} = 1$

This chiral theory was shown to have $W_{0,0}^{non-per.}(N_{3/2} = 1) = 0$; [24] perturbing it by a tree-level superpotential, $W_{tree} = gU$, where U is given in (3.4), may lead to dynamical supersymmetry breaking [24].

10 $b_1 = 0$

There are five cases with $b_1 = 0$: $N_f = 6$, or $N_A = 1$, $N_f = 4$, or $N_A = N_f = 2$, or $N_A = 3$, or $N_{3/2} = N_f = 1$. These theories have vanishing one-loop beta-functions in either conformal or infra-red free beta-functions and, therefore, will possess extra structure.

10.1 $N_f = 6, N_A = N_{3/2} = 0$

This theory is a particular case of $SU(N_c)$ with $N_f = 3N_c$; the electric theory is free in the infra-red [1].¹³

10.2 $N_f = 4, N_A = 1, N_{3/2} = 0$

In this case, the superpotential in (3.2) reads

$$W_{4,1} = -4 \frac{(\text{Pf} X)^{\frac{1}{4}}}{\Lambda^{\frac{b_1}{4}}} \Gamma^{\frac{1}{2}} + \tilde{m} M + \frac{1}{2} \text{Tr} m X + \frac{1}{\sqrt{2}} \text{Tr} \lambda Z. \quad (10.1)$$

Here m, X are antisymmetric 8×8 matrices, λ, Z are symmetric 8×8 matrices, Γ is given in eq. (7.6) and

$$\Lambda^{b_1} \equiv 16\alpha(\tau_0)^{1/2}(\det \lambda)^{-1/2}, \quad (10.2)$$

where $\alpha(\tau_0)$ will be presented in eq. (10.4). Let us discuss some properties of this theory:

- The equations $\partial W = 0$ can be re-organized into the singularity conditions of an elliptic curve (7.7) (and some other equations), where the coefficients a, b, c are [11, 12]

$$a = \frac{1}{\beta^2} \left\{ 2 \frac{\alpha + 1}{\alpha - 1} M + \frac{8}{\beta^2} \frac{\alpha}{(\alpha - 1)^2} \text{Tr}(\mu^2) \right\},$$

¹³ A related fact is that (unlike the $N_A = 1, N_f = 4$ case, considered next) in the (would be) superpotential, $W_{6,0} = -4\Lambda^{-b_1/4}(\text{Pf} X)^{1/4} + \frac{1}{2} \text{Tr} m X$, it is impossible to construct the matching “ $\Lambda^{b_1} \equiv f(\tau_0)$ ”, where τ_0 is the non-Abelian gauge coupling constant, in a way that respects the global symmetries.

$$\begin{aligned}
b &= \frac{1}{\beta^4} \left\{ -16 \frac{\alpha}{(\alpha-1)^2} M^2 + \frac{32}{\beta^2} \frac{\alpha(\alpha+1)}{(\alpha-1)^3} M \text{Tr}(\mu^2) \right. \\
&\quad \left. - \frac{8}{\beta^4} \frac{\alpha}{(\alpha-1)^2} [(\text{Tr}(\mu^2))^2 - 2\text{Tr}(\mu^4)] + \frac{4}{\beta^4} \frac{(\alpha+1)\Lambda^{b_1}}{(\alpha-1)^2} \text{Pf}m \right\}, \\
c &= \frac{1}{\beta^6} \left\{ -32 \frac{\alpha(\alpha+1)}{(\alpha-1)^3} M^3 + \frac{32}{\beta^2} \frac{\alpha(\alpha+1)^2}{(\alpha-1)^4} M^2 \text{Tr}(\mu^2) \right. \\
&\quad + M \left[-\frac{16}{\beta^4} \frac{\alpha(\alpha+1)}{(\alpha-1)^3} ((\text{Tr}(\mu^2))^2 - 2\text{Tr}(\mu^4)) + \frac{32}{\beta^4} \frac{\alpha\Lambda^{b_1}}{(\alpha-1)^3} \text{Pf}m \right] \\
&\quad \left. - \frac{32}{\beta^6} \frac{\alpha}{(\alpha-1)^2} [\text{Tr}(\mu^2)\text{Tr}(\mu^4) - \frac{1}{6}(\text{Tr}(\mu^2))^3 - \frac{4}{3}\text{Tr}(\mu^6)] \right\}. \quad (10.3)
\end{aligned}$$

Here $\mu = \lambda^{-1}m$ and α, β are functions of τ_0 , the non-Abelian gauge coupling constant; comparison with ref. [7] gives

$$\alpha(\tau_0) \equiv \frac{“\Lambda^{2b_1}”}{256} \det \lambda = \left(\frac{\theta_2^2 - \theta_3^2}{\theta_2^2 + \theta_3^2} \right)^2, \quad \beta(\tau_0) = \frac{\sqrt{2}}{\theta_2 \theta_3}, \quad (10.4)$$

where

$$\theta_2(\tau_0) = \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i \tau_0 n^2}, \quad \theta_3(\tau_0) = \sum_{n \in \mathbb{Z}} e^{\pi i \tau_0 n^2}, \quad \tau_0 = \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}. \quad (10.5)$$

In eq. (10.3) we have shifted the quantum field M to

$$M \rightarrow \beta^2 M - \alpha \text{Tr} \mu^2 / (\alpha - 1). \quad (10.6)$$

- The parameter x , in the elliptic curve (7.7), is given in terms of the composite field:

$$x \equiv \frac{1}{\beta^4} \left[\Gamma - \frac{4\alpha}{(\alpha-1)^2} \text{Tr} \mu^2 \right]. \quad (10.7)$$

- $W_{4,1}$ has $2+N_f = 6$ *quantum vacua*, corresponding to the 6 singularities of the elliptic curve (7.7), (10.3); these are the vacua of the theory in the *Higgs-confinement phase*.
- As before, from the phase transition points to the *Coulomb branch*, we conclude that the elliptic curve defines the *effective Abelian coupling*,

$\tau(M, \Lambda, m, \lambda)$, for arbitrary bare masses and Yukawa couplings. On the subspace of bare parameters, where the theory has $N = 2$ supersymmetry, the result in eq. (10.3) coincides with the result in [7] for $N_f = 4$.

- The discussion in the end of sections 7.2, 8.2 and 9.2 is relevant here too.
- We can get the results for $N_A = 1$, $N_f < 4$, by integrating out flavors.
- In all the $N_A = 1$, $N_f \neq 0$ cases we *derived* the result that τ is a section of an $SL(2, \mathbf{Z})$ bundle over the moduli space and over the parameters space of bare masses and Yukawa couplings (since τ is a modular parameter of a torus).

10.3 $N_f = 2$, $N_A = 2$, $N_{3/2} = 0$

It was argued that this theory is infra-red free [12].¹⁴

10.4 $N_f = 0$, $N_A = 3$, $N_{3/2} = 0$

In this case, the superpotential in eq. (3.2) reads

$$W_{0,3} = -4 \frac{(\det M)^{\frac{1}{2}}}{\Lambda^{\frac{b_1}{4}}} + \text{Tr} \tilde{m} M. \quad (10.8)$$

Here \tilde{m} , M are 3×3 symmetric matrices. The superpotential (10.8) equals to the tree-level superpotential, $W_{tree} = \lambda \det \Phi$, where, schematically, $\det \Phi \sim \epsilon \Phi \Phi \Phi \sim (\det M)^{1/2}$ is the (antisymmetric) coupling of the three triplets, Φ_α . This result coincides with the one derived in [23]. Therefore, we identify the matching “ $\Lambda^{-b_1/4} \equiv \lambda f(\tau_0)$ ”, which respects the global symmetries. In the massless case, this theory flows to an $N = 4$ supersymmetric Yang-Mills fixed point.

¹⁴A related fact is that (unlike the $N_A = 1$, $N_f = 4$ case) it is impossible to construct a matching, “ $\Lambda^{b_1} \equiv \alpha(\tau_0) f(\lambda^\alpha)$ ”, in a way that respects the global symmetries.

10.5 $N_f = 1, N_A = 0, N_{3/2} = 1$

It was argued that this theory is infra-red free [11].

11 More Results

We have summarized some old and new results in $N = 1$ supersymmetric $SU(2)$ gauge theories, and generalizations to other groups. More new results, which were derived in [12, 25, 22] along the lines of section 3, were presented in this meeting by E. Rabinovici. In particular, in ref. [25], we discussed how the structure of massless monopoles in supersymmetric theories with a Coulomb phase can be obtained from effective superpotentials for a phase with a confined photon. The technique was illustrated for $SU(N_c)$ in [25], and results for other gauge groups were reported this meeting and will appear in [22].

Finally, let us remark on the interplay between the properties of supersymmetric gauge theories and the properties of strings and extended objects. Much progress was made in understanding gauge dynamics by applying some “string theory intuition,” by taking the large Planck-mass limit, or by using probes in non-perturbative string backgrounds. Conversely, much progress was made in understanding non-perturbative string dynamics by applying some “gauge theory intuition.” The field develops fast, and an updated list of references would be enormous and will require a new update very soon.

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